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ON THE PRODUCT OF GÂTEAUX DIFFERENTIABILITY LOCALLY CONVEX SPACES [1]

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Abstract A locally convex space is said to be a Gâteaux differentiability space (GDS) provided every continuous convex function defined on a nonempty convex open subset D of the space is densely Gâteaux differentiable in D.This paper shows that the product of a GDS and a family of separable Fréchet spaces is a GDS,and that the product of a GDS and an arbitrary locally convex space endowed with the weak topology is a GDS.

Key words Convex function, locally convex space, Gâteaux differentiability space

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1 Introduction

The study of the differentiability properties of convex functions on infinite dimensional spaces has continued for almost seventy years since Mazur's intriguing 1933 work[1]. Based on Asplund's contribution[2], Namioka and Phelps[3] established the concept of (weak) Asplund spaces (those Banach spaces on which every continuous convex function is Fréchet (Gâteaux) differentiable on a dense G_δ subset). Since then, many achievements have been attained in this field (see e.g.,[4-7]). But little about the properties of weak Asplund spaces is known (see e.g,[8]).

The G_δ property is automatic for Fréchet differentiability, but known examples (see e.g., [9-10]) show that for Gâteaux differentiability, it is definitely an additional requirement. Therefore, in 1979 Larman and Phelps[11] introduced the notion of the Gâteaux differentiability space (GDS, see also Definition 1.2) and gave a characterization for GDS[11]. In the late eighties of the last century, Fabian[7, Proposition 6.5] showed that the product of a GDS and the real line is a GDS. The new important development in this filed is Cheng and Fabian's 2001 work[12] (see also Theorem 3.2). After 90's of the last century, people made many efforts for investigating differentiability behavior of convex functions and its applications in general Banach spaces and locally convex spaces (see e.g., [13-22]). The aim of this paper is to extend Cheng-Fabian's theorem[12] to locally convex spaces.

Now we recall some definitions which will be used in the sequel.

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(X,τ) will always be a locally convex space and $(X,\tau)^*$ its dual. We simply write X and X^* for (X,τ) and $(X,\tau)^*$ in the sequel.

Definition 1.1[7] A convex function f defined on a nonempty open convex subset D of X is said to be Gâteaux differentiable at $x \in D$, provided there exists $x^* \in X^*$ such that

$$\lim_{t\to 0^+}\left[\frac{f(x+ty)-f(x)}{t}-\langle x^*,y\rangle\right]=0,$$

for each $y \in X$.

Definition 1.2[14] A locally convex space X is called a Gâteaux differentiability space (GDS) if every continuous convex function defined on a nonempty convex open subset D of X is densely Gâteaux differentiable in D.

It is easy to see that if X is a GDS and (linear) isomorphic to Y, then Y is also a GDS. This paper mainly shows the following theorems.

Theorem 1.3 Let X be a Gâteaux differentiability space and Y be the product $\prod_{\alpha\in\mathcal{A}} E_\alpha$ of a family of separable Fréchet spaces (i.e., complete metrizable locally convex spaces). Then $X\times Y$ is also a Gâteaux differentiability space.

Theorem 1.4 Let X be a Gâteaux differentiability space and (Y,ω) be an arbitrary locally convex space endowed with the weak topology. Then $X\times(Y,\omega)$ is also a Gâteaux differentiability space.

Before starting the proof of the main theorems, we require some preparations.

2 Gâteaux Differentiability of Convex Functions

In this section we will show some results related to Gâteaux differentiability of convex functions on the locally convex space X.

Definition 2.1 Suppose that f is a continuous convex function defined on a nonempty open convex subset D of X and that A is a subset of X. We say that f is Gâteaux differentiable relative to A at $x\in D$, if there exists $x^*\in X^*$ such that

$$\lim_{t\to 0^+}\left[\frac{f(x+ty)-f(x)}{t}-\langle x^*,y\rangle\right]=0,$$

for each $y\in A$, or equivalently,

$$\lim_{t\to 0^+}\frac{f(x+ty)+f(x-ty)-2f(x)}{t}=0,$$

for each $y\in A$.

Definition 2.2[13] i) A real valued function f defined on a nonempty subset D of X is said to be locally Lipschitzian on D if there is a continuous seminorm p such that for each point $x_0\in D$ there exist a neighborhood U of x_0 and $L>0$ satisfying

$$|f(y)-f(x)|\le Lp(y-x),$$

for all $x,y\in U$;

ii) the function f is said to be Lipschitzian on D if it is locally Lipschitzian on D and L can be chosen to be independent of x_0.

Proposition 2.3[13,Prop.1.2] Suppose that f is a convex function defined on a nonempty convex subset D of X. If f is continuous at some point of D, then it is locally Lipschitzian on D.

Proposition 2.4 Suppose that f is a continuous convex function on a nonempty convex open subset D of X and A is a subset of X. If f is Gâteaux differentiable relative to A at $x \in D$ and span$A$ is dense in X, then f is Gâteaux differentiable at x.

Proof Since f is Gâteaux differentiable relative to A at x, we have

$$\lim_{t\to 0^+} \frac{f(x+ty)+f(x-ty)-2f(x)}{t} = 0, \tag{1}$$

for each $y \in A$. We need only show that (1) is true for all $y \in X$.

Let $C = \mathrm{co}(A\bigcup(-A))$. Then span $A = \bigcup_{\lambda>0}\lambda C$. We first show that f is Gâteaux differentiable relative to C at x. Indeed, each point y of C can be expressed as $y = \Sigma_{i=1}^n \lambda_i x_i$, where the λ_i's are non-negative real numbers such that $\Sigma_{i=1}^n \lambda_i = 1$, and $x_i \in A\cup(-A)$. By (1), for $\varepsilon > 0$, there exist $s_i > 0 (i = 1, 2, \cdots, n)$ such that

$$\frac{f(x+tx_i)+f(x-tx_i)-2f(x)}{t} < \varepsilon,$$

for all $t \in (0, s_i)$. Let $\delta = \min\{s_i\}_{i=1}^n$. Then for $t \in (0,\delta)$, we have

$$0 \le \frac{f(x+ty)+f(x-ty)-2f(x)}{t} \le \frac{\Sigma_{i=1}^n \lambda_i[f(x+tx_i)+f(x-tx_i)-2f(x)]}{t} < \varepsilon.$$

Thus, f is Gâteaux differentiable relative to span$A$ at x.

Next, we show that f is Gâteaux differentiable at x. If not, then there exists $y \in X$ such that

$$\frac{f(x+ty)+f(x-ty)-2f(x)}{t} \ge \varepsilon, \tag{2}$$

for some $\varepsilon > 0$ and all $t > 0$. Let $\{y_\alpha : \alpha \in \Lambda\}$ be a net in span A, which converges to the point y in X. Then

$$\frac{f(x+ty)+f(x-ty)-2f(x)}{t} = \frac{f(x+ty_\alpha)+f(x-ty_\alpha)-2f(x)}{t}$$
$$+\frac{f(x+ty)-f(x+ty_\alpha)}{t} + \frac{f(x-ty)-f(x-ty_\alpha)}{t}. \tag{3}$$

By Proposition 2.3, f is locally Lipschitzian at x, that is, there exist a continuous seminorm p, a neighborhood U of x in D and some $L > 0$ such that

$$|f(x+y)-f(x+z)| \le Lp(y-z),$$

for all $y, z \in U - x$. Take $\alpha \in \Lambda$ such that $p(y - y_\alpha) < \varepsilon/4L$. Then for this α and sufficiently small $t > 0$, we have

$$\left|\frac{f(x+ty_\alpha)+f(x-ty_\alpha)-2f(x)}{t}\right| < \frac{\varepsilon}{2}, \tag{4}$$

and

$$\left|\frac{f(x\pm ty)-f(x\pm ty_\alpha)}{t}\right| < \frac{\varepsilon}{4}. \tag{5}$$

From (2) to (5), it follows that $\varepsilon < \varepsilon$, a contradiction.

Proposition 2.5[14,Prop.2.9] Suppose that f is a continuous convex function on a nonempty open convex subset D of X. Then there is a continuous seminorm p such that for each $x \in D$ there exist a p-Lipschitz convex function g and a neighborhood U of x in D such that $f = g$ in U.

Proposition 2.6 A locally convex space X is a GDS if and only if every continuous convex function f on X is Gâteaux differentiable at some point of X.

Proof The necessity is plain. We concentrate on the converse.

Sufficiency If not, then there exist a nonempty convex open subset D and a continuous convex function f defined on D, such that the set of points where f is Gâteaux differentiable is not dense in D. Without loss of generality, we can assume that f is nowhere Gâteaux differentiable in D. By Proposition 2.5, for each $x_0 \in D$ there exist a continuous seminorm p, a p-Lipschitz convex function g on X and a neighborhood U of x_0 in D with $f = g$ in U. We assume that $x_0 = 0$(If necesary, we can translate both D and g). Define $g_m(x) = g(x/m), \forall x \in X$. Then g_m is a p-Lipschitz convex function on X and is nowhere Gâteaux differentiable in mU. Let $h = \Sigma_{m=1}^{\infty} 2^{-m} g_m$. Then h is a locally bounded convex function. That is, h is a continuous convex function on X which is nowhere Gâteaux differentiable in X, a contradiction.

3 Proof of the Main Theorems

We start with the following lemma.

Lemma 3.1 Let Y be a separable Fréchet space. Then there exist a dense subspace Y_0 of Y and a norm p on Y_0 such that (Y_0, p) is a separable Banach space.

Proof Since Y is a separable Fréchet space (i.e, separable metrizable locally convex space), there exists a sequence $\{x_n\}$ in Y with $x_n \to 0$ such that span$\{x_n\}$ is dense in Y. Let $C = \mathrm{co}\{\pm x_m\}$. Then C is a symmetric bounded convex set with $Y_1 \equiv \mathrm{span}\{x_n\} = \cup_{\lambda>0}\lambda C$. Let p be the Minkowski functional generated by C on Y_1. Clearly, (Y_1, p) is a separable normed space, and the norm topology on Y_1 is stronger than the original locally convex topology on Y restricted to Y_1. Let (Y_0, p) be the completion of (Y_1, p). We see that (Y_0, p) is a separable Banach space which is dense in Y in the original topology of Y.

We recall the following main theorem in[12], which is also true when X is a Gâteaux differentiable locally convex space.

Theorem 3.2[12] Let X be a Gâteaux differentiability space and Y be an arbitrary separable Banach space. Then $X \times Y$ is also a Gâteaux differentiability space.

Theorem 3.3 Let X be a Gâteaux differentiability space and Y be an arbitrary separable Fréchet space. Then $X \times Y$ is also a Gâteaux differentiability space.

Proof Let $f : X \times Y \to \mathbf{R}$ be a continuous convex function. By Proposition 2.6, we need only find a point in $X \times Y$ where f is Gâteaux differentiable.

By Lemma 3.1, there exist a dense subspace Y_0 of Y and a norm p on Y_0 such that (Y_0, p) is a separable Banach space. Since the topology of (Y_0, p) is stronger than the original topology of Y_0, f is also continuous on $X \times (Y_0, p)$. Then by Theorem 3.2, there is a point $(x_0, y_0) \in X \times (Y_0, p)$ such that

$$\lim_{t \to 0^+} \frac{f(x_0 + tx, y_0 + ty) + f(x_0 - tx, y_0 - ty) - 2f(x_0, y_0)}{t} = 0,$$

for each $(x,y) \in X \times Y_0$. That is, f is Gâteaux differentiable relative to $X \times Y_0$ at (x_0, y_0). Note that $X \times Y_0$ is dense in $X \times Y$. By Proposition 2.4, f is Gâteaux differentiable at $(x_0, y_0) \in X \times Y$.

Lemma 3.4 Suppose that f is a bounded convex function on a linear space X and $\mathrm{dom} f = X$. Then $f \equiv$ const. on X.

Proof If not, then there exist $x_1, x_2 \in X$ with $f(x_1) < f(x_2)$. We assume that $x_1 = 0$ and $f(x_1) = 0$ (if necessary we can translate f). Then by convexity of f,

$$f(x_2) = f\left(\frac{\lambda - 1}{\lambda} \cdot 0 + \frac{1}{\lambda} \cdot \lambda x_2\right) \le \frac{1}{\lambda} f(\lambda x_2), \quad \forall \lambda > 1.$$

That is, $f(\lambda x_2) \ge \lambda f(x_2)$, which implies that f is unbounded on X, a contradiction.

Now we restate the main theorems in Section 1.

Theorem 1.3 Let X be a Gâteaux differentiability space and Y be the product $\prod_{\alpha \in \mathcal{A}} E_\alpha$ of a family of separable Fréchet spaces. Then $X \times Y$ is also a Gâteaux differentiability space.

Proof Let $f: X \times Y \to \mathbf{R}$ be a continuous convex function. We need only show that f is Gâteaux differentiable at some point of $X \times Y$.

Since f is continuous, it is locally bounded. Thus, there exist a finite set $F \subset \mathcal{A}$ and some neighborhoods U and U_α of the origins 0 in X and E_α respectively such that f is bounded on $U \times \prod_{\alpha \in F} U_\alpha \times \prod_{\alpha \in \mathcal{A} \setminus F} E_\alpha$. By Lemma 3.4, for each fixed (x,y) in $U \times \prod_{\alpha \in F} U_\alpha$, $f(x,y,\cdot)$ is a constant on $\prod_{\alpha \in \mathcal{A} \setminus F} E_\alpha$. Let $g(x,y) = f(x,y,\cdot)$ on $U \times \prod_{\alpha \in F} U_\alpha$. Then g is a continuous convex function on $U \times \prod_{\alpha \in F} U_\alpha$. Since $X \times \prod_{\alpha \in F} E_\alpha$ is again a Gâteaux differentiability space, g is Gâteaux differentiable at some point $(x_0, y_0) \in U \times \prod_{\alpha \in F} U_\alpha$. So for each (x,y,z) in $X \times \prod_{\alpha \in F} E_\alpha \times \prod_{\alpha \in \mathcal{A} \setminus F} E_\alpha$, we have

$$\begin{aligned} & \frac{f(x_0 + tx, y_0 + ty, tz) + f(x_0 - tx, y_0 - ty, -tz) - 2f(x_0, y_0, 0)}{t} \\ = & \frac{g(x_0 + tx, y_0 + ty) + g(x_0 - tx, y_0 - ty) - 2g(x_0, y_0)}{t} \\ & \longrightarrow 0 \quad (\text{as } t \to 0^+). \end{aligned}$$

That is, f is Gâteaux differentiable at $(x_0, y_0, 0)$, which completes the proof.

Theorem 1.4 Let X be a Gâteaux differentiability space and (Y, ω) be an arbitrary locally convex space endowed with the weak topology. Then $X \times (Y, \omega)$ is also a Gâteaux differentiability space.

Proof Let $f: X \times (Y, \omega) \to \mathbf{R}$ be a continuous convex function. We need only show that f is Gâteaux differentiable at some point of $X \times (Y, \omega)$.

Since f is continuous, it is locally bounded. Thus, there exist some neighborhoods U and $V = \{x \in Y : |\langle x_i^*, x\rangle| < \varepsilon, x_i^* \in Y^* \setminus \{0\}, i = 1, 2, \cdots, n\}$ of the origins 0 in X and (Y, ω) respectively such that f is bounded on $U \times V$. Let $H = \cap_{i=1}^n \ker x_i^*$. We may assume that the x_i^*'s are linearly independent. Then Y is isomorphic to $\mathbf{R}^n \times H$. We can assume $Y = \mathbf{R}^n \times H$. So that $V = M \times H$, where M is a neighborhood of the origin 0 in $\mathbf{R}^n$. By Lemma 3.4, for each fixed (x,y) in $U \times M$, $f(x,y,\cdot)$ is a constant on H. Let $g(x,y) = f(x,y,\cdot)$ on $U \times M$. Then g is a continuous convex function on $U \times M$. Since $X \times \mathbf{R}^n$ is again a Gâteaux differentiability space, g is Gâteaux differentiable at some point $(x_0. y_0) \in U \times M$. So for each (x,y,z) in $X \times \mathbf{R}^n \times H$, we have

$$\begin{aligned}&\frac{f(x_0+tx,y_0+ty,tz)+f(x_0-tx,y_0-ty,-tz)-2f(x_0,y_0,0)}{t}\\ =&\frac{g(x_0+tx,y_0+ty)+g(x_0-tx,y_0-ty)-2g(x_0,y_0)}{t}\\ &\longrightarrow 0 \quad (\text{as } t\to 0^+).\end{aligned}$$

And this says that f is Gâteaux differentiable at $(x_0, y_0, 0)$, which completes the proof.

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